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A variational study of directed polymers in disordered media with short-range correlations

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Abstract. A variational method that allows for replica-symmetry breaking is applied to directed polymers in an $(N+1)$-dimensional disordered medium. The noise studied here has Gaussian correlations, i.e. it is short-ranged. In dimensions $N<2$, the variational scheme yields only a strong-coupling phase and anomalous diffusion; while in dimensions $N>2$ it shows weak- and strong-coupling phases but no anomalous diffusion. Comparisons are made with the results of Mézard and Parisi for noise with power-law correlations.

The sluggish rate of progress on strongly disordered systems attests to the difficulty of the problem. Some headway has been made over the past several years in the area of manifolds subject to a random potential. These may represent the most tractable models in which disorder plays a crucial role. Furthermore, they have been tied to various physical phenomena. For instance, the one-dimensional incarnation, a directed polymer in a random environment [1–12], has been related to domain walls in two-dimensional disordered ferromagnets [1], surface growth [13], and randomly stirred fluids [14].

By a 'directed' polymer, one means a polymer or random walker that always proceeds in a positive direction along a given coordinate in an $(N+1)$-dimensional space. The effect of the disorder is manifested by the walker's accumulating a series of random weights, one associated with each site through which it passes. The walker's attempt to maximize the benefit of the random potential results in far greater stretching along the axes transverse to the directed axis than the purely entropic spreading of its non-random counterpart. In the non-random case, the transverse fluctuations are diffusive, i.e. scale as $\langle\omega^2(\ell)\rangle \sim \ell$ (where ℓ measures the distance along the directed axis and ω that along the transverse axes); whereas in the random case, the transverse fluctuations are superdiffusive, i.e. scale as $\overline{\langle\omega^2(\ell)\rangle} \sim \ell^{2\nu}$ (for large ℓ) where $\nu \geqslant \frac{1}{2}$ (where $\overline{f(x)}$ indicates averaging $f(x)$ over realizations of the disorder).

This non-trivial behaviour is deeply rooted in the quenched nature of the disorder—as is the difficulty in the analysis thereof. Thus, it seems natural that the issues and techniques familiar from another problem featuring quenched randomness, namely spin glasses, have resurfaced here. Derrida and Spohn [6] made the connection to spin glasses explicit when they uncovered a mapping between directed polymers on a Cayley tree and the random energy model, which is in some sense 'the simplest spin glass' [14].

The replica approach, another technique from the study of spin glasses, has made its way into the study of manifolds in disordered media. Early on, Kardar [4] used it to suggest that $\overline{\langle\omega^2(\ell)\rangle} \sim \ell^{4/3}$ for $(1+1)$-dimensional directed polymers with delta-function correlated disorder. More recently, Mézard and Parisi (MP) [11] applied a variational version of the replica method to the general problem of random manifolds pinned by quenched random impurities. This variational approach has also found applications in the study of (non-directed) polymers in a random media [16], protein-folding [17], disordered vortex lattices [18], the random-field Ising model [19], and interference effects in variable-range hopping [20, 21].

In their extensive treatment of manifolds [11], MP included the case of directed polymers with delta-function correlations by invoking dimensional analysis, that is, by considering power-law correlations that scale in the same way as the delta functions: for instance, $1/|x|$ would replace $\delta^{(1)}(x)$. In the present work, we apply the variational replica method to directed polymers with noise correlations that fall off like a Gaussian $\sim e^{-\omega^2}$—a genuine short-range behaviour.

To recap briefly, MP [11] studied a very generic Hamiltonian for D-dimensional manifolds in an $(N+D)$-dimensional space:

$$H[\omega] = \frac{1}{2}\int dx \sum_{\mu=1}^{D}\left(\frac{\partial\omega}{\partial x_\mu}\right)^2 + \int dx\, V(x, \omega(x)) \tag{1}$$

where $\omega(x)$ is an N-component vector field and x is a D-dimensional vector. The potential $V(x,\omega)$ is a random variable with zero mean and correlation

$$\overline{V(x,\omega)V(x',\omega')} = -N\delta^{(D)}(x-x') f\left(\frac{[\omega-\omega']^2}{N}\right). \tag{2}$$

The scaling with N was introduced to facilitate the large-N limit in which the variational approach used is expected to become exact; it will be maintained here for purposes of comparison.

MP considered noise correlations of the power-law form:

$$f_{\rm pl}\left(\frac{\omega^2}{N}\right) \sim \frac{g}{2(1-\gamma)}\left(\frac{\omega^2}{N}\right)^{1-\gamma} \tag{3}$$

for $\omega^2 \gg 1$. Under these conditions, their calculations split quite naturally into two cases: noise with 'long-range' correlation (either $D \geqslant 2$ or $D < 2$ and $\gamma < 2/(2-D)$) and noise with short-range correlation ($D < 2$ and $\gamma > 2/(2-D)$). Again, we note that MP only considered directed polymers with delta-function correlations indirectly by relating them to power-law correlations with the same scaling, in particular, by setting $\gamma = 1 + N/2$. One might find it somewhat dissatisfying that this prescription places directed polymers with delta-function correlations in $(1+1)$ and $(2+1)$ dimensions into their 'long-range' category. Another possible source of discontentment is MPs treatment of the 'short-range' case, in which they expanded the short-range behaviour of the correlation in a power series, truncating it after only two terms. The present calculation employs a bona fide short-range form for the correlation and requires no such truncation. It will be shown to reproduce the main results of [11] for directed polymers.

In the directed-polymer case ($D = 1$), the partition function corresponding to $H[\omega]$ (1) is a path integral

$$Z(\omega, x) = \int_{\{0,0\}}^{\{\omega,x\}} \mathcal{D}\omega\, \exp\Big\{-\beta H[\omega]\Big\} \tag{4a}$$

(where β is the reciprocal temperature) which can be shown to obey the equation:

$$\frac{\partial Z(\omega,x)}{\partial x} = -\mathcal{H}(\omega)\, Z(\omega,x) \tag{4b}$$

where $\mathcal{H}$ is the (random) operator:

$$\mathcal{H}(\omega) = -\frac{1}{2\beta}\sum_{i=1}^{N}\frac{\partial^2}{\partial\omega_i^2} + \beta V(x, \omega(x)) \,. \tag{4c}$$

But it can be usual first to replicate the system n times and average over the disorder; one then obtains a path integral which obeys

$$\frac{\partial \overline{Z^n(\{\omega_a\}, x))}}{\partial x} = -\mathcal{H}_n(\{\omega_a\})\overline{Z^n(\{\omega_a\}, x)} \tag{5a}$$

with the following (non-random) n-body Hamiltonian:

$$\mathcal{H}_n(\{\omega_a\}) = -\frac{1}{2\beta}\sum_{i=1}^{N}\sum_{a=1}^{n}\frac{\partial^2}{\partial\omega_{ai}^2} + \frac{\beta^2}{2}\sum_{a\neq b}^{n} N f\left(\frac{1}{N}(\omega_a - \omega_b)^2\right) \tag{5b}$$

with attractive interactions given by f (as defined in (2)).

The interesting exact solutions to such an n-body Schrödinger-like equation have already been exploited [4, 8], and standard perturbative approaches have been shown to yield unphysical results [11]. Hence, we shall proceed here with a variational scheme of the Rayleigh–Ritz type. That is, we shall study the ground-state energy of $\mathcal{H}_n(\{\omega_a\})$ using the following variational wavefunction:

$$\Psi = \mathcal{N}\exp\left\{-\tfrac{1}{2}\sum_{a,b}^{n}\sum_{i=1}^{N}\hat{m}_{a,b}\omega_{ia}\omega_{ib}\right\}. \tag{6}$$

This method of searching for the variational ground state of an operator Hamiltonian is convenient when considering one-dimensional manifolds ($D = 1$) [12, 17, 20, 21] and differs slightly from the approach taken in [11] where higher-dimensional manifolds are considered as well. The results, however, do not differ.

Keeping in mind the eventual $n \to 0$ limit and the possible emergence of replica-symmetry breaking (RSB) therein, a Gaussian form was chosen for ease of calculation prior to that limit. The matrix $\hat{m}$ of variational parameters is chosen as an $n \times n$ hierarchical matrix with K-step breaking (as introduced by Parisi [22]). In the one-step ($K = 1$) breaking scenario, one can envision breaking the replicas into groups with the couplings between 'particles' belonging to the same group differing from the couplings between 'particles' belonging to different groups. Next, two-step breaking divides the groups into subgroups, and so on, leading to the hierarchical (ultrametric) structure. Eventually, the limits $K \to \infty$ and $n \to 0$ are to be taken. In addition to offering one more variational parameter (than the replica-symmetric version), it is hoped that the variational scheme with RSB can better mimic situations in which two or more diverging paths have nearly degenerate energies, including those with the very lowest energies [9].

With the Gaussian wavefunction, calculating $\langle\mathcal{H}_n\rangle_\Psi$ requires computing $\mathrm{Tr}(\hat{m})$ (which arises from the expectation of the kinetic-energy portion of $\mathcal{H}_n$) and certain functions of $(m_{a,a}^{-1} + m_{b,b}^{-1} - 2m_{a,b}^{-1})$ (which emerge from the interaction terms). To be more specific, a Taylor expansion and applying Wick's theorem to an interaction term lead to [11]:

$$\begin{aligned}\left\langle f\left[\frac{(\omega_a - \omega_b)^2}{N}\right]\right\rangle_\Psi &= \sum_j \frac{f^{(j)}}{j!N^j}\langle(\omega_a - \omega_b)^{2j}\rangle_\Psi \\ &= \sum_j \frac{f^{(j)}}{j!N^j}\frac{(N+2j-2)!!}{(N-2)!!}\frac{(m_{a,a}^{-1} + m_{b,b}^{-1} - 2m_{a,b}^{-1})^j}{2^j} \\ &= \hat{f}\left[\frac{m_{a,a}^{-1} + m_{b,b}^{-1} - 2m_{a,b}^{-1}}{2}\right]\end{aligned} \tag{7a}$$

where

$$\hat{f}(z) = \frac{1}{\Gamma(N/2)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{N/2-1}\, \mathrm{e}^{-\alpha} f\left(\frac{2\alpha z}{N}\right). \tag{7b}$$

We are currently interested in the case in which the correlations have a Gaussian form:

$$f_{\mathrm{g}}\left(\frac{(\omega_a-\omega_b)^2}{N}\right) = -\left(\frac{\mu^2}{\pi N}\right)^{N/2} \exp\left\{\frac{-\mu^2}{N}(\omega_a-\omega_b)^2\right\} \tag{8a}$$

where $N^{\frac{1}{2}}/\mu$ serves as a correlation length and $f_g(z)$ is normalized here so that it approaches a delta function in the $\mu \to \infty$ limit. The corresponding $\hat{f}$ is:

$$\hat{f}_{\mathrm{g}}(z) = -\left(\frac{\mu^2}{\pi N}\right)^{N/2} \left[1 + \frac{2\mu^2 z}{N}\right]^{-\frac{N}{2}}. \tag{8b}$$

For power-law correlations (3), the same procedure yields:

$$\hat{f}_{\mathrm{pl}}(z) = \frac{g\Gamma(1+N/2-\gamma)}{2\,(1-\gamma)\,\Gamma(N/2)}\left(\frac{2z}{N}\right)^{1-\gamma} \tag{9}$$

provided $\gamma \leqslant 1 + N/2$. Note that $\hat{f}_g(z)$ and $\hat{f}_{\mathrm{pl}}(z)$ have the same large-z behaviour when $\gamma = 1 + N/2$, furnishing evidence for the validity of MPs dimensional-analysis approach within this scheme. (Setting $\gamma = 1 + N/2$ requires some small-z regularization; otherwise, the coefficient of $\hat{f}_{\mathrm{pl}}$ would diverge.)

The K-step hierarchical matrix $\hat{m}$ has $K+2$ parameters $\{\tilde{a}, a_0, a_1, \ldots, a_K\}$ which take on the form $[\tilde{a}, a(u)]$, where $a(u)$ is a function on the interval $[n, 1]$, in the $n \to 0$ limit [22]. References [21] showed how the eigenvalues of $\hat{m}$, which take on a similar form $[\tilde{\lambda}, \lambda(u)]$, are related to $[\tilde{a}, a(u)]$. In terms of the eigenvalues, $\langle \mathcal{H}_n \rangle_\Psi$ becomes

$$\frac{\langle \mathcal{H} \rangle_\Psi}{Nn} = \frac{1}{4\beta} \int_1^n \frac{\mathrm{d}u}{u^2} \lambda(u) + \frac{1}{4\beta n}\tilde{\lambda} + \beta^2 \int_1^n \mathrm{d}u\, \hat{f}[Q(u)] \tag{10a}$$

where

$$Q(u) = \int_1^u \frac{\mathrm{d}v}{v^2} \lambda^{-1}(v) + \frac{1}{u}\lambda^{-1}(u). \tag{10b}$$

The first two terms in (10a) correspond to $\mathrm{Tr}(\hat{m})$, i.e. the sum over the eigenvalues. Note that $\mathrm{d}u/u^2$ is roughly the degeneracy of $\lambda(u)$.

Now comes the variation. One obtains solutions for the 'best' $\lambda(u)$ through the following procedure. First, take a functional derivative of (10a) with respect to $\lambda(u)$ and set it equal to zero; this yields the stationarity equation:

$$\lambda^2(u) = 4\beta^3 \left\{ \int_u^n \mathrm{d}v\, \hat{f}'[Q(v)] + u \hat{f}'[Q(u)] \right\} \tag{11}$$

where $\hat{f}' = \partial \hat{f}/\partial Q$. As a step towards finding an equation which is local in u (i.e. no integrals over u), take a derivative with respect to u, which results in

$$2\lambda(u)\lambda_u(u) = -4\beta^3 \hat{f}''[Q(u)]\lambda^{-2}(u)\lambda_u(u) \tag{12a}$$

where $\lambda_u = \partial\lambda/\partial u$. This result implies that either $\lambda_u(u) = 0$ or

$$2^{-1}\beta^{-3}\lambda^3(u) = -\hat{f}''[Q(u)]\,. \tag{12b}$$

To pursue the latter case, put in the desired form of $f''(x)$,

$$\hat{f}''_{\rm g}(x) = -\frac{\mu^4(N+2)}{N}\left(\frac{\mu^2}{\pi N}\right)^{N/2}\left[1+\frac{2\mu^2 x}{N}\right]^{-(N+4)/2} \tag{13a}$$

and for simplicity let

$$s = \frac{\mu^4(N+2)}{N}\left(\frac{\mu^2}{\pi N}\right)^{N/2} \tag{13b}$$

and

$$t = \frac{2\mu^2}{N}\,. \tag{13c}$$

These substitutions produce

$$2^{-1}\beta^{-3}\lambda^3(u) = s\Big[1 + t\,Q(u)\Big]^{-(N+4)/2} \tag{14a}$$

which upon inversion becomes

$$\left[2^{-1}\beta^{-3}s^{-1}\lambda^3(u)\right]^{-2/(N+4)} = 1 + t\,Q(u) \tag{14b}$$

where $Q(u)$, is given by (10*b*). Now, equation (14*b*) is still non-local in u, so take another derivative with respect to u, giving

$$-\left(\frac{6}{N+4}\right)\left(2\beta^3 s\right)^{2/(N+4)}\lambda^{-(N+10)/(N+4)}(u)\lambda_u(u) = -\frac{t}{u}\lambda^{-2}(u)\lambda_u(u)\,. \tag{15}$$

The solution of (15) is either $\lambda_u(u) = 0$ or

$$\lambda(u) = \left(2\beta^3 s\right)^{2/(2-N)}\left(\frac{6u}{(N+4)t}\right)^{(N+4)/(2-N)}\,. \tag{16}$$

Note that (16) and $\lambda(u)$ = constant (which corresponds to $\lambda_u(u) = 0$) are merely the possible local solutions. One must piece together a function on the entire interval $[n, 1]$, consisting of these local solutions, which satisfies the (non-local) stationarity equation (11).

The pattern of RSB found by MP in their 'long-range' solution fuses together a power-law behaviour (similar to (16)) for $n \leqslant u < u_{\rm c}$ and a constant for $u_{\rm c} < u \leqslant 1$. MP demonstrated that this small-u, power-law dependence is intimately connected to the property of superdiffusion: $\overline{\langle\omega^2(\ell)\rangle} \sim \ell^{2\nu}$ with $\nu > \frac{1}{2}$. In particular, they found for ν the 'Flory' exponents [5] (e.g. $\nu = \frac{3}{5}$ for directed polymers in 1 + 1 dimensions, instead of the exact result $\nu = \frac{2}{3}$ [3,4]).

For the Gaussian problem, we will search for solutions of a similar nature. Note, however, that for $N > 2$, the power law is not well behaved for small u in the $n \to 0$ limit—then we will seek solutions like those derived by MP for their short-range case, which consist of two constants (i.e. one-step breaking). Hence, the present calculation also splits into two parts: $N < 2$, where one finds the full-breaking, power-law solution for small u and which displays anomalous diffusion; and $N > 2$, where one finds the one-step breaking solution, a phase transition (see below), but no anomalous diffusion.

First, for the case $N < 2$, let us assume a solution of the form

$$\lambda(u) = \begin{cases} \left(2\beta^3 s\right)^{2/(2-N)}\left[\dfrac{6u}{(N+4)t}\right]^{(N+4)/(2-N)} & \text{if } n < u < u_c \\ \left(2\beta^3 s\right)^{2/(2-N)}\left[\dfrac{6u_c}{(N+4)t}\right]^{(N+4)/(2-N)} & \text{if } u_c < u < 1 \end{cases} \tag{17}$$

then one can determine u_c by considering (14) at $u = 1$ (bypassing some tedious integration). This procedure yields the following equation for u_c:

$$\frac{N}{\mu^2}\left[\frac{N(N+2)\beta^3}{4\pi^{N/2}}\right]^{2/(2-N)}\left(\frac{6u_c}{N+4}\right)^{6/(2-N)} + \frac{6u_c}{(N+4)} - 1 = 0\,. \tag{18}$$

This solution, equations (17) and (18), was also obtained numerically by iteratively solving a discretized version of the stationarity equation (11), giving credence to the assumptions made in (17).

Note that here u_c is β-dependent. This feature differs from MPs solution for power-law correlations, where u_c is a constant [11]. With a β-dependent u_c, one ought to check for solutions with u_c lying outside the interval [0, 1], as that might indicate some kind of a phase transition, but u_c always lies within this range for $N < 2$. When $\beta = 0$ or $\mu = \infty$ (the delta-function limit), the first term in (18) is zero, and one finds $u_c = (N+4)/6$, which is precisely the constant found by MP for power-law correlations (if $\gamma = 1 + N/2$). Then, as β increases (or μ decreases) u_c approaches though does not reach zero, so that $0 < u < (N+4)/6$. For finite μ and large β, one finds $u_c \sim \beta^{-1}$. This property distinguishes predictions concerning the large-β behaviour of quantities (such as the free energy) made by the Gaussian and power-law calculations.

For $N > 2$, we look for a solution with one-step breaking of the form

$$\lambda(u) = \begin{cases} 0 & \text{if } 0 < u < u_c \\ A & \text{if } u_c < u < 0\,. \end{cases} \tag{19}$$

Substituting this solution into the Hamiltonian (10a) yields

$$\frac{\langle\mathcal{H}\rangle_\Psi}{Nn} = \frac{1}{4\beta}\left(1 - \frac{1}{u_c}\right)A + \beta^2(1-u_c)\left(\frac{\mu^2}{\pi N}\right)^{N/2}\left(1 + \frac{t}{A}\right)^{-N/2}. \tag{20}$$

Note that when $\lambda(u) = 0$, $Q(u) = \infty$. Next, taking derivatives with respect to A and u_c and setting them equal to zero results in

$$u_c = \frac{A^2}{2N\beta^3 t}\left(\frac{\pi N}{\mu^2}\right)^{N/2}\left(1 + \frac{t}{A}\right)^{(N+2)/2} \tag{21}$$

and

$$u_c^2 = \frac{A}{4\beta^3}\left(\frac{\pi N}{\mu^2}\right)^{N/2}\left(1 + \frac{t}{A}\right)^{N/2} \tag{22}$$

i.e. the stationarity equations.

Eliminating A leads to

$$1 = \frac{t}{8\beta^3 u_c^3}\left(\frac{\pi N^2}{\mu^2}\right)^{N/2}(N - 2u_c)^{1-N/2}. \tag{23}$$

Again, we check for situations in which u_c *lies outside the interval* $[n, 1]$. *This time we* find some. In fact, we find β_c, the critical β, simply by setting $u_c = 1$ in (23):

$$\beta_c = \frac{t^{1/3}}{2}\left(\frac{\pi N^2}{\mu^2}\right)^{N/6}(N-2)^{(2-N)/6}. \tag{24}$$

For high temperatures, $\beta < \beta_c$, the solution of (23) has $u_c > 1$; so the solution is then $\lambda(u) = 0$ for the entire interval $[n, 1]$—the (trivial) replica-symmetric solution. Here, too, the assumptions made in (19) were tested using the discretized version of (11).

The presence of a transition to a weak-coupling phase for $N > 2$ is in accord with the known behaviour of directed polymers [6]. As mentioned earlier, in their consideration of short-range correlations, MP expanded $\hat{f}(z) \approx f_0 + f_1 z + \ldots$ for small z, keeping only the first two terms in their explicit calculations. In the Gaussian case, no such approximation was deemed necessary, and yet the same basic results were obtained. Hence, it indicates that MPs results are not purely an artifact of the truncation procedure.

In summary, we have applied a variational replica method to $(N+1)$-dimensional directed polymers with Gaussian-correlated noise. The calculation divided quite naturally into two regimes. In the first, $N < 2$, the variational solution displayed full replica-symmetry breaking with a power-law behaviour for small u. This feature is associated with superdiffusion, and in particular the 'Flory' exponents for the transverse fluctuations are found. Similar behaviour was observed by MP [11] in their so-called 'long-range' solution for noise with power-law correlations. Another trait shared by the Gaussian-noise and power-law-noise calculations [11] is the lack of a transition to a weak-coupling (high temperature) phase for $N < 2$. On the other hand, one distinction between the two is their dependence on the inverse temperature β, especially for large β. In the second regime, $N > 2$, the variational solution displayed a phase with no replica-symmetry breaking for high temperatures and a phase with one-step replica-symmetry breaking at lower temperatures, but no anomalous diffusion.

The variational scheme manages to uncover the phase transition where it is expected ($N > 2$) and the absence of a transition where it is expected ($N < 2$); that is, the correct phase diagram is obtained. It predicts anomalous diffusion for $N < 2$; however, it fails to find the correct exponent ν for the transverse fluctuations. Moreover, it predicts no superdiffusion in the strong-coupling phase for $N > 2$, as is anticipated from the simulations [10]. One scheme for furthering calculations involves an expansion (in $1/N$) around the solution at $N = \infty$ which is proposed to be exact [23, 24]. Another conceivable (albeit less controlled) approach would be to improve the Gaussian assumption for the variational wavefunction, perhaps by using a Lanczos-type approach [24] to go beyond the basic Rayleigh–Ritz method. It is hoped that the present work has resolved a few of the nagging issues found in MP. However, the more important issues remain unresolved and more work is required before this chapter can be closed.

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References

[1] Huse D A and Henley C L 1985 *Phys. Rev. Lett.* **54** 2708
[2] Kardar M and Zhang Y-C 1987 *Phys. Rev. Lett.* **58** 2087
[3] Huse D A, Henley C L and Fisher D S 1985 *Phys. Rev. Lett.* **55** 2924
[4] Kardar M 1987 *Nucl. Phys.* B **290** 582
[5] Nattermann T 1985 *J. Phys. C: Solid State Phys.* **18** 6661
Kardar M1987 *J. Appl. Phys.* **61** 3601
[6] Derrida B and Spohn H 1988 *J. Stat. Phys.* **51** 817
[7] Medina E, Hwa T, Kardar M and Zhang Y-C 1989 *Phys. Rev.* A **39** 3053
Imbrie J Z and Spencer T 1988 *J. Stat. Phys.* **52** 609
[8] Parisi G 1990 *Rend. Acad. Naz. Lincei* **XI-1** 3
[9] Mézard M 1990 *J. Physique* **51** 1831
[10] Kim J M, Moore M A and Bray A J 1991 *Phys. Rev.* A **44** 2345
[11] Mézard M and Parisi G 1991 *J. Physique* I **1** 809; 1990 *J. Phys. A: Math. Gen.* **23** L1299
[12] Mézard M and Parisi G 1992 *J. Phys. A: Math. Gen.* **25** 4521
[13] Kardar M, Parisi G and Zhang Y-C 1986 *Phys. Rev. Lett.* **56** 889
[14] Forster D, Nelson D and Stephen M 1977 *Phys. Rev.* A **16** 732
[15] Derrida B 1981 *Phys. Rev.* B **24** 2613
Gross D J and Mézard M 1984 *Nuc. Phys.* B **240** 431
[16] Edwards S F and Muthukumar M 1988 *J. Chem Phys.* **89** 2435
[17] Shakhnovich E I and Gutin A M 1989 *J. Phys. A: Math. Gen.* **22** 1647
[18] Bouchaud J-P, Mézard M and Yedidia J S 1991 *Phys. Rev. Lett.* **67** 3840; 1992 *Phys. Rev.* B **66** 16686
Giamarchi T and Le Doussal P 1993 Elastic theory of pinned flux lattices *Preprint*
[19] Mézard M and Young A P 1992 *Europhys. Lett.* **18** 653
[20] Goldschmidt Y Y and Blum T 1992 *J. Physique* I **2** 1607
[21] Blum T and Goldschmidt Y Y 1992 *J. Phys. A: Math. Gen.* **25** 6517
[22] Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[23] Goldschmidt Y Y 1993 *Nucl. Phys.* B **393** 507; 1993 Manifolds in random media: Beyond the variational approximation *Preprint*
[24] Cook J and Derrida 1988 *Europhys. Lett.* **10** 195; 1990 *J. Phys. A: Math. Gen.* **23** 1523
[25] Dagotto E and Moreo A 1985 *Phys. Rev.* D **31** 865